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LETTER TO THE EDITOR

Fractal structures derivable from the generalisations of the Pascal triangle

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Abstract. Generalisations, of order $K \geqslant 2$, of the Pascal triangle are used to construct generalised Pascal–Sierpinski gaskets of orders (K, $L \geqslant 2$). It is shown that all such gaskets are self-affine fractals, but when $K = 2$ and L is prime then the gasket is rigorously self-similar and possesses a similarity dimension. The evolutionary morphology of the gaskets of orders (K, L prime) bears a resemblance to the growth of pyrolitic graphite films and other material structures.

In an earlier report (Holter *et al* 1986) we described a new family of planar fractals derivable from the famous Pascal triangle and named the members of this family as the Pascal–Sierpinski gaskets (PSG). It was shown that the PSG of prime order are rigorously self-similar, for which fractal (similarity) dimensions can easily be deduced, while those of non-prime order are merely self-affine: various fractal measures can be obtained for them experimentally or numerically (Lakhtakia *et al* 1986a, 1987). Moreover, the well known Sierpinski gasket (Mandlebrot 1983) turns out to be a PSG as well.

Further generalisation of the PSG is afforded by the extension (Philippou *et al* 1985) of the original Pascal triangle (Uspenskii 1974). Given an integer $K \geqslant 2$, let ${}^{(K)}P_{n,m}$ be defined as a rectangular array on the integral indices n and m by the relations

$$ {}^{(K)}P_{n,m} = 0 \qquad m > n(K-1),\ n \geqslant 0 \qquad (1a)$$

$$ {}^{(K)}P_{n,m} = \sum_{i \in \{0,m\}} {}^{(K)}P_{n-1,m-i} \qquad 0 \leqslant m < K,\ n \geqslant 1 \qquad (1b)$$

$$ {}^{(K)}P_{n,m} = \sum_{i \in \{0,K-1\}} {}^{(K)}P_{n-1,m-i} \qquad K \leqslant m \leqslant n(K-1),\ n \geqslant 1 \qquad (1c)$$

in which the recursive process of calculation begins with the seed

$$ {}^{(K)}P_{0,0} = 1. \qquad (1d)$$

If attention is focused solely on the range $0 \leqslant m \leqslant n(K-1)$, $\forall n \geqslant 0$, then the array of numbers thus obtained constitutes a generalised Pascal triangle of order K. It should be noted that $K = 2$ for the ordinary Pascal triangle.

The generalised Pascal triangles give rise to many fractal structures in the same manner that the PSG were generated from the ordinary Pascal triangle. Given another integer $L \geqslant 2$, let the array ${}^{(K,L)}Q_{n,m}$ be constructed from ${}^{(K)}P_{n,m}$ via the relation

$$ {}^{(K,L)}Q_{n,m} = \text{mod}\{{}^{(K)}P_{n,m}, L\}. \qquad (2)$$

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Then the generalised Pascal–Sierpinski gasket (GPSG) of order (K, L) is defined as the array ${}^{(K,L)}G_{n,m}$, where

$$ {}^{(K,L)}G_{n,m} = 0 \qquad \text{if} \quad {}^{(K,L)}Q_{n,m} = 0 \tag{3a} $$

$$ {}^{(K,L)}G_{n,m} = 1 \qquad \text{if} \quad {}^{(K,L)}Q_{n,m} \neq 0. \tag{3b} $$

It should be noted that the GPSG of order (K, L) is not the same as that of the order (L, K). A couple of GPSG are shown in figure 1.

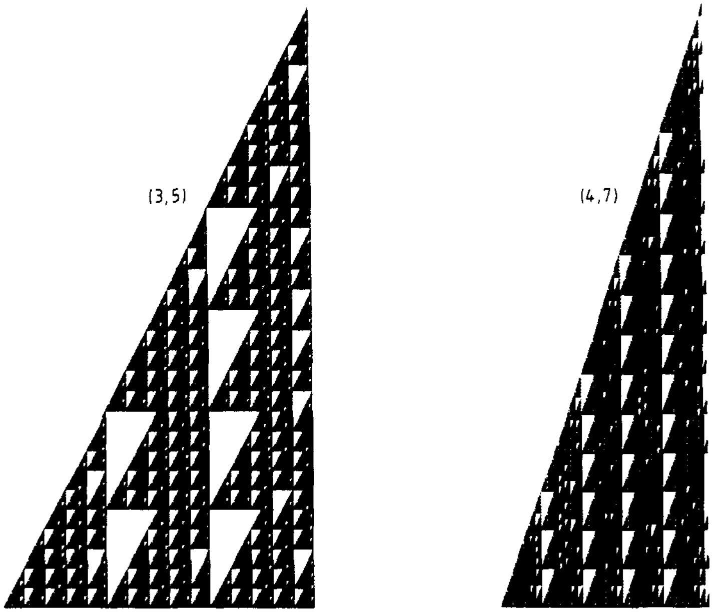

Figure 1. The generalised Pascal–Sierpinski gaskets of orders (3, 5) and (4, 7).

From visual inspection of some one hundred GPSG, it became clear that, in general, they are not rigorously self-similar; the exceptions are the GPSG of order $(2, L)$ provided L is prime, and in which case the similarity dimension turns out to be

$$ d_{(2,L)} = \log[L(L+1)/2]/\log[L] \qquad L = 2, 3, 5, 7, 11, 13, \ldots. \tag{4} $$

Thus, the mass–radius dimension has to be computed as per the procedure described by us elsewhere in detail (Holter *et al* 1986). Shown in figure 2 are several plots of $\log[M(n)]$ against $\log[n]$, where the mass function is defined as the sum

$$ M(n) = \sum_{i\in\{0,n\}} \sum_{m\in\{0,i(K-1)\}} {}^{(K,L)}G_{n,m}. \tag{5} $$

The broken lines marked n in this figure are the plots which are obtained provided the zeros of ${}^{(K,L)}G_{n,m}$ are all replaced by 1, and hence correspond to the Euclidean dimension of two. It is to be noted from these graphs that the mass–radius dimensions of the GPSG exceed unity but fall short of the Euclidean dimension. Generally speaking, for a fixed K, the GPSG of higher L possesses the higher mass–radius dimension. There appears to be a minimum limit for this dimension, namely $\log 3/\log 2$ which is the similarity dimension of the Sierpinski gasket. Coupled with the fact that evidence of self-similarity was obtained only for the PSG of prime order, it can be stated that the

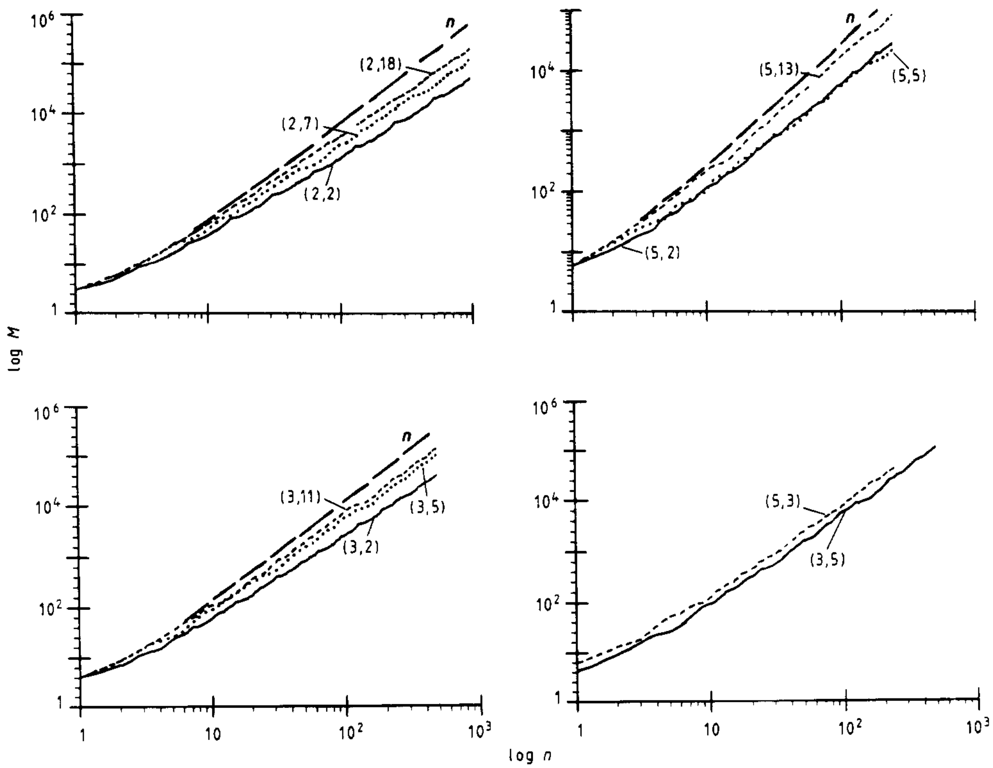

Figure 2. Plots of the mass function against the row index n for several GPSG. The broken curves marked n would be obtained if the zeros of the GPSG were all replaced by 1, and their topological dimensions equal their Euclidean dimensions of two.

vast majority of the GPSG are merely self-affine (Mandelbrot 1985, Lakhtakia *et al* 1986b). That, however, does not detract from their fractal nature: indeed, they may be among the few well ordered and deterministic examples of fractals which are not self-similar.

If the PSG of prime order are self-similar, the natural question to ask is why the GPSG of orders ($K \geq 3$, L prime) are not? (Further discussion in this letter is confined to prime L.) In order to answer this question, we explored the function ${}^{(K,L)}G_{n,m}$ and came up with an interesting observation. The row $n = L^p$, $p \geq 1$, contains a total of $[1 + L^p(K-1)]$ lattice sites on which there are $(K-1)$ groups of zeros (*nulls*) ensconced. Each group contains $(L^p - 1)$ nulls, and is bounded on either side by a 1(seed). Thus, in the row $n = L^p$, there are always K seeds and $(K-1)(L^p-1)$ nulls. Since this grouping keeps on recurring every L^pth row, it would appear that the proper fractal scale is logarithmic with base L for the pertinent GPSG regardless of the specific value of K.

The L^pth row should be regarded, in the manner of Huyghens' principle, as a reseeding row for the GPSG. Each seed on this row evolves out into a triangular gasketlet made up of $L^p(L-1)$ rows. Down its L^pth row, each gasketlet would also contain $[1+(L^p-1)(K-1)]$ sites, but

$$[1+(L^p-1)(K-1)] \geq [1+(L^p-1)] \tag{6}$$

the equality holding only if $K = 2$. Consequently, the additional gasketlets will interfere with each other and reduce the number of sites to be occupied with a value of 1, unless $K = 2$. The mass function $M(L^p)$ does not scale with L, therefore, if $K > 2$. Hence, this interference is responsible for the self-affinity of the GPSG of orders ($K > 2$, L prime); conversely, the absence of this interference makes the GPSG of orders (2, L prime) self-similar.

From looking at the evolution of the GPSG, several conclusions on these structures can be drawn. These are enumerated as follows.

(*a*) Self-similarity is destroyed by interference even in the presence of a well established logarithmic scale.

(*b*) As the size (p) increases, voids of different sizes do not all increase in number at the same rate.

(*c*) Reseeding takes place quickly when p is small.

(*d*) Competition starts in the vicinity of the L^pth row.

The two concepts of reseeding and interference (i.e. competition) between structures are central to developing an understanding of thin-film growth under low adatom mobility conditions (Messier 1986). For such conditions, a thin film has a void network with an internal boundary structure which is apparently fractal (Messier and Yehoda 1985). Thus, similarities between the cross-sectional morphology of vapour-deposited films and the present GPSG may have the same basis, despite the fact that in thin-film growth reseeding occurs randomly and continuously. Depending upon the film deposition conditions, the nucleation density, the competition for resulting cone-growth survival and the renucleation frequency are all interrelated and can account for the wide range in film properties and morphology: from the lower density and rough top surfaces (cauliflower-like) to dense smooth morphologies. Quantitative deterministic fractal models, such as the present example, may hold the key to quantitative preparation-property relations for thin films.

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